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SHORT COMMUNICATIONS

Transformation law and trace formula on theta series under Siegel modular group*

LU Hongwen and ZHOU Haigang**

(Department of Applied Mathematics, Tongji University, Shanghai 200092, China)

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Abstract The purpose of this paper is to discuss symplectic transformation laws on theta series and give an explicit formula for trace of the symplectic operator

Keywords: theta series, symplectic transformation law, trace formula.

Theta series plays an important role in number theory and also in the theory of modular forms. In 1978, Shimura[1] established a close relation between Jacobi forms and theta series. In 1985, Eichler and Zagier[2] developed systematically the theory of Jacobi forms. Later, Skoruppa and Zagier[3] studied the trace formula for Jacobi forms. Li[4] studied the trace formula for Jacobi forms of general degree. In his formula, the trace of the operator on the theta series is of importance. But he did not compute the trace formula explicitly. The computation of the transformation law of theta series is interest though it is not easy to perform. Many mathematicians contributed greatly to this problem. However, the definition of theta series here is slightly different from the usual symplectic theta series they studied. The purpose of this paper is to give some symplectic transformation laws on theta series and to obtain the trace formula of symplectic operator.

Throughout this paper, l means a positive even integer and $\boldsymbol{S}$ a positive definite, half integral and symmetric matrix of degree l. For a square matrix $\boldsymbol{X}$, we use the symbol $e\{\boldsymbol{X}\}$ as an abbreviation of $\exp(2\pi i\,\mathrm{tr}\boldsymbol{X})$.

Let $Sp(n, \mathbb{Z})$ denote the Siegel modular group, that is, symplectic group over $\mathbb{Z}$, and $\mathbb{H}_n$ the Siegel upper half-plane of degree n, $\mathbb{H}_n := \{\tau \in Sym(n, \mathbb{C}) \mid \mathrm{Im}\tau > 0\}$.

For any $\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix} \in Sp(n, \mathbb{Z})$, the action of $Sp(n, \mathbb{Z})$ on $\mathbb{H}_n \times \mathbb{C}^{(l,n)}$ is defined by

$$\boldsymbol{M}(\tau, z) := (\boldsymbol{M}\langle\tau\rangle, z(\boldsymbol{C}\tau + \boldsymbol{D})^{-1}),$$

where $\boldsymbol{M}\langle\tau\rangle := (\boldsymbol{A}\tau + \boldsymbol{B})(\boldsymbol{C}\tau + \boldsymbol{D})^{-1}$.

Suppose k is an integer. For a function $\phi(\tau, z)$ on $\mathbb{H}_n \times \mathbb{C}^{(l,n)}$, we define an operator $|_{k,S}$ by setting

$$(\phi\,|_{k,S}\boldsymbol{M})(\tau, z) := J_{k,S}(\boldsymbol{M}(\tau, z))\phi(\boldsymbol{M}(\tau, z)),$$

where $J_{k,S}(\boldsymbol{M}, (\tau, z)) := \det(\boldsymbol{C}\tau + \boldsymbol{D})^{-k} e\{-\boldsymbol{S}z(\boldsymbol{C}\tau + \boldsymbol{D})^{-1}\boldsymbol{C}^t z\}$, which is a factor of automorphy.

Definition 1. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Q}^{(l,n)}$, the theta series with character $(\boldsymbol{a}, \boldsymbol{b})$ and index $\boldsymbol{S}$ is defined by

$$\theta_{S,a,b}(\tau, z) := \sum_{u \in \mathbb{Z}^{(l,n)}} e\{\boldsymbol{S}(\boldsymbol{u} + \boldsymbol{a})\tau^t(\boldsymbol{u} + \boldsymbol{a}) + 2(\boldsymbol{u} + \boldsymbol{a})^t(z + \boldsymbol{b})\}. \tag{1}$$

The right hand side converges absolutely and uniformly on any compact subset of $\mathbb{H}_n \times \mathbb{C}^{(l,n)}$. If $\boldsymbol{b} = 0$, $\theta_{S,a,b}(\tau, z)$ is also denoted by $\theta_{S,a}(\tau, z)$.

Definition 2. For a fixed point $\tau \in \mathbb{H}_n$, we define $\mathcal{T}_S(\tau)$ to be the vector space of all holomorphic functions ϕ: $\mathbb{C}^{(l,n)} \to \mathbb{C}$ satisfying

$$\phi(z + \boldsymbol{u}\tau + \boldsymbol{v}) = e\{\boldsymbol{S}(\boldsymbol{u}\tau^t\boldsymbol{u} + 2\boldsymbol{u}^t z)\}\phi(z)$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{Z}^{(l,n)}$.

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** To whom correspondence should be addressed. E-mail: haigangz@sina com

For any $M \in Sp(n, \mathbb{Z})$, we define an operator $U_S(M)$ on the vector space $\mathcal{T}_S(\tau)$ by

$$\theta \mid U_S(M) := \theta \mid_{\frac{l}{2}, S} M.$$

Proposition 1. Let $\mathcal{N}$ be a complete system of representatives of the cosets $(2S)^{-1}\mathbb{Z}^{(l,n)}/\mathbb{Z}^{(l,n)}$. Then the functions $\{\theta_{S,a}(\tau, z) \mid a \in \mathcal{N}\}$ form a basis of $\mathcal{T}_S(\tau)$. In particular, the dimension of $\mathcal{T}_S(\tau)$ is $(\det 2S)^n$.

Proof. See Ref. [5].

Lemma 1. The Siegel modular group $Sp(n, \mathbb{Z})$ is generated by the following matrices: $U(A) := \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}$, where $A \in SL(n, \mathbb{Z})$, $T(B) := \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}$, where $B \in Sym(n, \mathbb{Z})$ and $J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Proof. See Theorem 3.6 in Ref. [6].

Lemma 2. Suppose $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z})$, rank $(C) = s$, $0 < s < n$, there exists V and $\tilde{V} \in SL(n, \mathbb{Z})$, such that $V^* C\tilde{V} = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $V^* D\tilde{V}^* = \begin{pmatrix} D_1 & 0 \\ 0 & I_{n-s} \end{pmatrix}$, where $C_1, D_1 \in \mathbb{Z}^{(s,s)}$, $\operatorname{rank}(C_1) = s$.

Proof. See Lemma 1.18 in Ref. [6].

Lemma 3. If $M = \begin{pmatrix} * & * \\ 0 & I_s \end{pmatrix} \in Sp(n, \mathbb{Z})$, then there exist matrices

$$\widetilde{M} = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & I_{n-s} & 0 & 0 \\ C_1 & 0 & D_1 & 0 \\ 0 & 0 & 0 & I_{n-s} \end{pmatrix}$$

and $W = \begin{pmatrix} I_s & 0 & 0 & {}^t v \\ u & I_{n-s} & v & w \\ 0 & 0 & I_s & -{}^t u \\ 0 & 0 & 0 & I_{n-s} \end{pmatrix} \in Sp(n, \mathbb{Z})$, such that $M = \widetilde{M}W$.

Proof. See Ref. [7].

For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z})$, according to the rank of C, $\operatorname{rank}(C) = s$, we consider M in three cases: $s = 0$, $s = n$ and $0 < s < n$. For the case of $0 < s < n$, in terms of Lemmas 2 and 3, it is sufficient to consider the simply form like $\widetilde{M}$.

In terms of the properties of Siegel modular group and theta series, we get our main results.

Theorem 1. Suppose $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in Sp(n, \mathbb{Z})$, and $a \in \mathcal{N}$, we have

$$\theta_{S,a} \mid U_S(M)(\tau, z) = e\{SaB^tA^ta\}\theta_{S,aA}. \quad (2)$$

Theorem 2. Suppose $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z})$, with $\det C \neq 0$, and $a \in \mathcal{N}$, we have

$$\theta_{S,a} \mid U_S(M)(\tau, z) = \sum_{b \in \mathcal{N}} C_{ab}\theta_{S,b}(\tau, z), \quad (3)$$

where

$$\begin{aligned} C_{ab} &= (\det 2S)^{-\frac{n}{2}}(\det iC)^{-\frac{l}{2}} \\ &\cdot e\{S(aAC^{-1t}a - 2aC^{-1t}b + bC^{-1}D^tb)\} \\ &\cdot \sum_{r \in \mathbb{Z}^{(l,n)}/(\mathbb{Z}^{(l,n)t}C)} e\{S(rAC^{-1t}r + 2rAC^{-1t}a \\ &- 2rC^{-1t}b)\}. \end{aligned}$$

Theorem 3. Suppose

$$M = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & I_{n-s} & 0 & 0 \\ C_1 & 0 & D_1 & 0 \\ 0 & 0 & 0 & I_{n-s} \end{pmatrix} \in Sp(n, \mathbb{Z}),$$

with $\det C_1 \neq 0$, and $a \in \mathcal{N}$. Let a_1 be the first s columns of matrix a, and a_2 be the rest part of a. For $\tilde{b} \in (2S)^{-1}\mathbb{Z}^{(l,s)}/\mathbb{Z}^{(l,s)}$, we set $b = (\tilde{b}, a_2)$. So we have

$$\theta_{S,a} \mid U_S(M)(\tau, z) = \sum_{\tilde{b} \in (2S)^{-1}\mathbb{Z}^{(l,s)}/\mathbb{Z}^{(l,s)}} C_{ab}\theta_{S,b}(\tau, z), \quad (4)$$

where

$$\begin{aligned} C_{ab} &= (\det 2S)^{-\frac{n}{2}}(\det iC_1)^{-\frac{s}{2}} \\ &\cdot e\{S(a_1A_1C_1^{-1t}a_1 - 2a_1C_1^{-1t}\tilde{b} + \tilde{b}C_1^{-1}D_1{}^t\tilde{b})\} \\ &\cdot \sum_{r \in \mathbb{Z}^{(l,s)}/(\mathbb{Z}^{(l,s)t}C_1)} e\{S(rA_1C_1^{-1t}r \\ &+ 2rvA_1C_1^{-1t}a_1 - 2rC_1^{-1t}\tilde{b})\}. \end{aligned}$$

From the above theorems we get the following corollaries.

Corollary 1. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z})$, we have

(i) when $\boldsymbol{C}=0$,

$$\mathrm{tr}U_{\boldsymbol{S}}(\boldsymbol{M}) = \sum_{\substack{\boldsymbol{a}\in\mathcal{N} \\ \boldsymbol{aA}=\boldsymbol{a}}} e\{2\boldsymbol{SaB}^t\boldsymbol{a}\}; \tag{5}$$

(ii) when $\det\boldsymbol{C}\neq 0$,

$$\begin{aligned}\mathrm{tr}U_{\boldsymbol{S}}(\boldsymbol{M}) &= (\det 2\boldsymbol{S})^{-\frac{n}{2}}(\det i\boldsymbol{C})^{-\frac{l}{2}} \\ &\cdot \sum_{\boldsymbol{q}\in\mathbb{Z}^{(l,n)}/(2\boldsymbol{S}\mathbb{Z}^{(l,n)})} e\Big\{\frac{1}{4}(\boldsymbol{q}(\boldsymbol{AC}^{-1} \\ &+ 2\boldsymbol{C}^{-1}+\boldsymbol{C}^{-1}\boldsymbol{D})^t\boldsymbol{q})\boldsymbol{S}^{-1}\Big\} \\ &\cdot \sum_{\boldsymbol{r}\in\mathbb{Z}^{(l,n)}/(\mathbb{Z}^{(l,n)t}\boldsymbol{C})} e\{\boldsymbol{SrAC}^{-1t}\boldsymbol{r} \\ &+ \boldsymbol{r}(\boldsymbol{AC}^{-1}-\boldsymbol{C}^{-1})^t\boldsymbol{q}\};\end{aligned} \tag{6}$$

(iii) when $\mathrm{rank}(\boldsymbol{C})=s$, $0<s<n$,

$$\boldsymbol{M}=\begin{pmatrix}\boldsymbol{A}_1 & 0 & \boldsymbol{B}_1 & 0 \\ 0 & \boldsymbol{I}_{n-s} & 0 & 0 \\ \boldsymbol{C}_1 & 0 & \boldsymbol{D}_1 & 0 \\ 0 & 0 & 0 & \boldsymbol{I}_{n-s}\end{pmatrix}\in Sp(n,\mathbb{Z}),$$

$$\begin{aligned}\mathrm{tr}U_{\boldsymbol{S}}(\boldsymbol{M}) &= (\det 2\boldsymbol{S})^{-\frac{s}{2}}(\det i\boldsymbol{C}_1)^{-\frac{l}{2}} \\ &\cdot \sum_{\boldsymbol{q}\in\mathbb{Z}^{(l,n)}/(2\boldsymbol{S}\mathbb{Z}^{(l,n)})} e\Big\{\frac{1}{4}(\boldsymbol{q}(\boldsymbol{A}_1\boldsymbol{C}_1^{-1} \\ &+ 2\boldsymbol{C}_1^{-1}+\boldsymbol{C}_1^{-1}\boldsymbol{D}_1)^t\boldsymbol{q})\boldsymbol{S}^{-1}\Big\} \\ &\cdot \sum_{\boldsymbol{r}\in\mathbb{Z}^{(l,s)}/(\mathbb{Z}^{(l,s)t}\boldsymbol{C}_1)} e\{\boldsymbol{SrA}_1\boldsymbol{C}_1^{-1t}\boldsymbol{r} \\ &+ r(\boldsymbol{A}_1\boldsymbol{C}_1^{-1}-\boldsymbol{C}_1^{-1})^t\boldsymbol{q}\}.\end{aligned} \tag{7}$$

For $\boldsymbol{A}$ an $n\times n$ symmetric matrix with rational entries and for $\boldsymbol{Q}$ an $m\times m$ symmetric integer matrix with even integers on the main diagonal, let $G(\boldsymbol{A},\boldsymbol{B})$ denote the following Gauss sum

$$G(\boldsymbol{A},2\boldsymbol{Q}) = d^{-mn}\sum_{\boldsymbol{L}\in\mathbb{Z}^{(m,n)}/(d\mathbb{Z}^{(l,n)})} q\{{}^t\boldsymbol{LQLA}\},$$

where d is any positive integer such that $d\boldsymbol{A}$ is an integer matrix.

Corollary 2. For $\boldsymbol{M}=\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D}\end{pmatrix}\in Sp(n,\mathbb{Z})$ with $\det\boldsymbol{C}=1$, n, l are even positive integers, we have

$$\begin{aligned}\mathrm{tr}U_{\boldsymbol{S}}(\boldsymbol{M}) = (\det 2\boldsymbol{S})^{\frac{n}{2}}G(2\boldsymbol{S},\boldsymbol{AC}^{-1}+\boldsymbol{C}^{-1} \\ +{}^t\boldsymbol{C}^{-1}+\boldsymbol{C}^{-1}\boldsymbol{D}).\end{aligned} \tag{8}$$

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